Electronic Journal of Qualitative Theory of Differential Equations
2017, No. **65**, 1–14; doi: 10.14232/ejqtde.2017.1.65 http://www.math.u-szeged.hu/ejqtde/

Existence of solutions for an age-structured insect population model with a larval stage

Božena Dorociaková ✉ [1], **Rudolf Olach**[1] and **Iveta Ilavská**[2]

[1]University of Žilina, Faculty of Mechanical Engineering, Department of Applied Mathematics
Univerzitná 1, Žilina 010 26, Slovakia
[2]University of Žilina, Faculty of Operation and Economics of Transport and Communications
Department of Quantitative Methods and Economic Informatics, Univerzitná 1, Žilina 010 26, Slovakia

Received 15 July 2016, appeared 23 September 2017
Communicated by Josef Diblík

Abstract. In this paper we particularly draw attention to the existence of solutions which describe the maturation rates of an age-structured insect population model. Such models commonly divide the population into immature and mature individuals. Immature individuals are defined as individuals of age less than some threshold age τ, while adults are individuals of age exceeding τ. The model is represented by the nonlinear neutral differential equation with variable coefficients. The conditions, which guarantee that the population size tends to a nonnegative constant or nonconstant function are also established.

Keywords: population models, age structure, neutral delay equation, existence, fixed point theorem.

2010 Mathematics Subject Classification: 34K13.

1 Introduction

In the paper [11] authors Gourley and Kuang study the age structured population model. Such model is suitable for describing the dynamics of an insect population with larval and adult phases. As example may be the periodical cicada. The life phases of cicadas and other types of insect are fully described in [11]. Gourley and Kuang start the derivation of model with equation

$$\frac{\partial u(t,a)}{\partial t} + \frac{\partial u(t,a)}{\partial a} = -\mu(a)u(t,a),$$

where t is time, a denotes age and $u(t,a)$ is the density of individuals of age a at time t, $\mu(a)$ is death rate. It is assumed that the population is divided into immature and mature individuals. Immature individuals are defined as individuals of age less than some threshold age τ, while adults are individuals of age exceeding τ. Then the numbers $I(t)$ and $M(t)$ of immature and mature individuals respectively, are given by

✉Corresponding author. Email: bozena.dorociakova@fstroj.uniza.sk

$$I(t) = \int_0^\tau u(t,a)\,da, \qquad M(t) = \int_\tau^\infty u(t,a)\,da.$$

It is supposed that the total number of adults $M(t)$ obeys the equation

$$\frac{dM(t)}{dt} = u(t,\tau) - d(M(t)). \tag{1.1}$$

The term $u(t,\tau)$ is the maturation rate and $d(M)$ is the adult mortality function which is strictly increasing in M and $d(0) = 0$. The authors in [11] consider the equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = -\mu u \tag{1.2}$$

with a constant linear death rate μ and

$$u(t,0) = \int_0^\infty b(a)u(t,a)\,da,$$

where $b(a) \geq 0$ is the birth rate function and the initial condition

$$u(0,a) = u_0(a) \geq 0, \qquad a \geq 0. \tag{1.3}$$

In [11] it is considered that

$$b(a) = b_0 + (b_1 - b_0)H(a-\tau) + b_2\delta(a-\tau),$$

where $H(a)$ is the Heaviside function and $\delta(a)$ the Dirac delta function. It is assumed that individuals of age less than τ produce b_0 eggs per unit time, those of age greater than τ produce b_1 eggs per unit time and each individual lays b_2 eggs on reaching maturation age τ.

For the adult population there is derived in [11] for $t \geq \tau$ the neutral delay differential equation

$$M'(t) = [b_2M'(t-\tau) + b_2d(M(t-\tau)) + b_0I(t-\tau) + b_1M(t-\tau)]e^{-\mu\tau} - d(M(t)).$$

For $0 \leq t \leq \tau$ and $u(t,\tau) = u_0(\tau - t)e^{-\mu t}$ the $M(t)$ is governed by the equation

$$M'(t) = u_0(\tau - t)e^{-\mu t} - d(M(t)), \qquad 0 \leq t \leq \tau.$$

Further in the paper the authors consider the case when $b_0 = 0$. For more details about model we refer readers to [11] and for related models see [3, 10, 12, 13]. Qualitative properties of differential equations with delay are studied, for example, in [1, 2, 4–8, 15].

Instead of constants b_1, b_2 in the function $b(a)$ we will consider positive bounded functions $b_1(M(t))$, $b_2(t)$, $t \geq 0$, since b_1 is dependent on the total numbers of mature. Then following [11] we get

$$b(t,a) = b_0 + (b_1(M(t)) - b_0)H(a-\tau) + b_2(t)\delta(a-\tau)$$

and for $u(t,0)$ it follows

$$\begin{aligned} u(t,0) &= \int_0^\infty b(t,a)u(t,a)\,da \\ &= \int_0^\infty [b_0 + (b_1(M(t)) - b_0)H(a-\tau) + b_2(t)\delta(a-\tau)]u(t,a)\,da \\ &= b_2(t)u(t,\tau) + b_0\int_0^\tau u(t,a)\,da + b_1(M(t))\int_\tau^\infty u(t,a)\,da. \end{aligned}$$

For $u(t,0)$ we obtain

$$u(t,0) = b_2(t)u(t,\tau) + b_0 I(t) + b_1(M(t))M(t). \tag{1.4}$$

The solution of (1.2) with respect to (1.3) and $u(t,0) = N(t)$ is given by

$$u(t,a) = \begin{cases} u_0(a-t)\exp(-\mu t), & 0 \le t < a, \\ N(t-a)\exp(-\mu a), & t > a. \end{cases}$$

For $t > \tau$ with regard to (1.4) we get

$$\begin{aligned} u(t,\tau) &= N(t-\tau)e^{-\mu\tau} = u(t-\tau,0)e^{-\mu\tau} \\ &= [b_2(t-\tau)u(t-\tau,\tau) + b_0 I(t-\tau) + b_1(M(t-\tau))M(t-\tau)]e^{-\mu\tau}. \end{aligned}$$

equation (1.1) implies that

$$u(t-\tau,\tau) = M'(t-\tau) + d(M(t-\tau)).$$

Thus we get

$$\begin{aligned} u(t,\tau) = \big[&b_2(t-\tau)M'(t-\tau) + b_2(t-\tau)d(M(t-\tau)) + b_0 I(t-\tau) \\ &+ b_1(M(t-\tau))M(t-\tau)\big]e^{-\mu\tau}. \end{aligned}$$

Then the equation (1.1) has the form

$$\begin{aligned} M'(t) = \big[&b_2(t-\tau)M'(t-\tau) + b_2(t-\tau)d(M(t-\tau)) + b_0 I(t-\tau) \\ &+ b_1(M(t-\tau))M(t-\tau)\big]e^{-\mu\tau} - d(M(t)), \qquad t \ge \tau, \end{aligned}$$

which is the neutral delay differential equation.

For $t \le \tau$, $u(t,\tau) = u_0(\tau - t)e^{-\mu t}$ and with respect to equation (1.1) we obtain

$$M'(t) = u_0(\tau-t)e^{-\mu t} - d(M(t)), \qquad 0 \le t \le \tau.$$

In the rest of this paper we will assume that $b_0 = 0$. Motivated by the discussion above, we will consider the differential equations

$$x'(t) = u_0(\tau-t)e^{-\mu t} - f(x(t)), \qquad 0 \le t \le \tau, \tag{1.5}$$

$$\begin{aligned} &\frac{d}{dt}[x(t) - e^{-\mu\tau}b_2(t-\tau)x(t-\tau)] \\ &\quad = e^{-\mu\tau}\big[b_1(x(t-\tau))x(t-\tau) \\ &\qquad - b_2'(t-\tau)x(t-\tau) + b_2(t-\tau)f(x(t-\tau))\big] - f(x(t)), \qquad t \ge \tau, \end{aligned} \tag{1.6}$$

where $u_0 \in C([0,\tau],[0,\infty))$, $\mu,\tau \in (0,\infty)$, $b_1 \in C((0,\infty),(0,\infty))$, $b_2 \in C^1([0,\infty),(0,\infty))$ are bounded functions, b_1 is nondecreasing, $b_2'(t) \le 0$, $f \in C((0,\infty),(0,\infty))$ is nondecreasing function.

It is reasonable to assume that the size of population is bounded. In this paper we consider a population which size is bounded by the functions $w(t)$, $v(t)$. The function $w(t)$, for example, can depend on food resources, seasonal conditions, the size of territory in which the population lives, etc. We will focus on the existence of positive solutions for the neutral differential equation (1.6), since this problem is not solved in [11]. The conditions which guarantee that the population size tends to nonnegative constant are also established. We also study the case when the size of population tends to nonconstant function. The main contribution of [11] is the Theorem 4. But this theorem we cannot apply to the Example 3.2.

The following fixed point theorem will be used to prove the main results in the next section.

Lemma 1.1 (see [9, 16] Krasnoselskii's fixed point theorem)**.** *Let X be a Banach space, let Ω be bounded closed convex subset of X, and let S_1, S_2 be maps of Ω into X such that $S_1x + S_2y \in \Omega$ for every pair $x, y \in \Omega$. If S_1 is a contraction and S_2 is completely continuous then the equation*

$$S_1x + S_2x = x$$

has a solution in Ω.

2 Existence theorems

The aim of the paper is to show the correctness of the model, that is, to show that the equation which represents the model, has a positive solution.

Theorem 2.1. *Suppose that*

$$b_2(t-\tau)e^{-\mu\tau} < 1, \qquad t \geq \tau, \tag{2.1}$$

and there exist bounded functions $v, w \in C^1([0,\infty),(0,\infty))$, constant $K \geq 0$ such that

$$v(t) \leq w(t), \qquad t \geq 0, \tag{2.2}$$

$$w(t) - w(\tau) - v(t) + v(\tau) \geq 0, \qquad 0 \leq t \leq \tau, \tag{2.3}$$

$$\begin{aligned}
&\frac{1}{v(t-\tau)}\bigg(v(t) - K + \int_t^\infty \Big(e^{-\mu\tau}\big[b_1(w(s-\tau))w(s-\tau) - b_2'(s-\tau)w(s-\tau) \\
&\qquad\qquad + b_2(s-\tau)f(w(s-\tau))\big] - f(v(s))\Big)\, ds\bigg) \\
&\quad \leq b_2(t-\tau)e^{-\mu\tau} \\
&\quad \leq \frac{1}{w(t-\tau)}\bigg(w(t) - K + \int_t^\infty \Big(e^{-\mu\tau}\big[b_1(v(s-\tau))v(s-\tau) - b_2'(s-\tau)v(s-\tau) \\
&\qquad\qquad\qquad + b_2(s-\tau)f(v(s-\tau))\big] - f(w(s))\Big)\, ds\bigg), \qquad t \geq \tau.
\end{aligned} \tag{2.4}$$

Then equation (1.6) has a positive solution which is bounded by functions v, w.

Proof. Let $C([0,\infty), R)$ be the Banach space of all continuous bounded functions with the norm $\|x\| = \sup_{t\geq 0}|x(t)|$. We define a closed, bounded and convex subset Ω of $C([0,\infty), R)$ as follows

$$\Omega = \{x = x(t) \in C([0,\infty), R) : v(t) \leq x(t) \leq w(t),\ t \geq 0\}.$$

We now define two maps S_1 and $S_2 : \Omega \to C([0,\infty), R)$ as follows

$$(S_1x)(t) = \begin{cases} e^{-\mu\tau}b_2(t-\tau)x(t-\tau) + K, & t \geq \tau, \\ (S_1x)(\tau), & 0 \leq t \leq \tau, \end{cases}$$

$$(S_2x)(t) = \begin{cases} -\int_t^\infty \Big(e^{-\mu\tau}\big[b_1(x(s-\tau))x(s-\tau) - b_2'(s-\tau)x(s-\tau) \\ \qquad\qquad + b_2(s-\tau)f(x(s-\tau))\big] - f(x(s))\Big)\,ds, \quad t \geq \tau, \\ (S_2x)(\tau) + w(t) - w(\tau), \quad 0 \leq t \leq \tau, \end{cases}$$

We will show that $S_1x + S_2y \in \Omega$ for any $x, y \in \Omega$. For $t \geq \tau$ and every $x, y \in \Omega$, applying (2.4) we derive

$$\begin{aligned} &(S_1x)(t) + (S_2y)(t) \\ &\quad = e^{-\mu\tau}b_2(t-\tau)x(t-\tau) + K \\ &\qquad - \int_t^\infty \Big(e^{-\mu\tau}\big[b_1(y(s-\tau))y(s-\tau) - b_2'(s-\tau)y(s-\tau) \\ &\qquad\qquad + b_2(s-\tau)f(y(s-\tau))\big] - f(y(s))\Big)\,ds \\ &\quad \leq e^{-\mu\tau}b_2(t-\tau)w(t-\tau) + K \\ &\qquad - \int_t^\infty \Big(e^{-\mu\tau}\big[b_1(v(s-\tau))v(s-\tau) - b_2'(s-\tau)v(s-\tau) \\ &\qquad\qquad + b_2(s-\tau)f(v(s-\tau))\big] - f(w(s))\Big)\,ds \leq w(t). \end{aligned} \tag{2.5}$$

For $t \in [0, \tau]$ using the inequality $(S_1x)(\tau) + (S_2y)(\tau) \leq w(\tau)$ we obtain

$$\begin{aligned} (S_1x)(t) + (S_2y)(t) &= (S_1x)(\tau) + (S_2y)(\tau) + w(t) - w(\tau) \\ &\leq w(\tau) + w(t) - w(\tau) = w(t). \end{aligned}$$

Furthermore using (2.4) for $t \geq \tau$ we get

$$\begin{aligned} (S_1x)(t) + (S_2y)(t) &\geq e^{-\mu\tau}b_2(t-\tau)v(t-\tau) + K \\ &\quad - \int_t^\infty \Big(e^{-\mu\tau}[b_1(w(s-\tau))w(s-\tau) - b_2'(s-\tau)w(s-\tau) \\ &\qquad + b_2(s-\tau)f(w(s-\tau))] - f(v(s))\Big)\,ds \geq v(t). \end{aligned} \tag{2.6}$$

Let $t \in [0, \tau]$. With regard to (2.3) we obtain

$$w(t) - w(\tau) + v(\tau) \geq v(t).$$

According to inequality above and $(S_1x)(\tau) + (S_2y)(\tau) \geq v(\tau)$, for $t \in [0, \tau]$ and any $x, y \in \Omega$ we get

$$\begin{aligned} (S_1x)(t) + (S_2y)(t) &= (S_1x)(\tau) + (S_2y)(\tau) + w(t) - w(\tau) \\ &\geq v(\tau) + w(t) - w(\tau) \geq v(t). \end{aligned}$$

Thus we have proved that $S_1x + S_2y \in \Omega$ for any $x, y \in \Omega$.

We will show that S_1 is a contraction mapping on Ω. For $x, y \in \Omega$ and $t \geq \tau$ we derive

$$|(S_1x)(t) - (S_1y)(t)| = e^{-\mu\tau}b_2(t-\tau)|x(t-\tau) - y(t-\tau)| \leq e^{-\mu\tau}b_2(t-\tau)\|x - y\|.$$

This yields that

$$\|S_1x - S_1y\| \leq e^{-\mu\tau}b_2(t-\tau)\|x - y\|.$$

Such inequality is also valid for $t \in [0, \tau]$. With regard to (2.1) S_1 is a contraction mapping on Ω.

We now show that S_2 is completely continuous. First we will show that S_2 is continuous. Let $x_k = x_k(t) \in \Omega$ be such that $x_k(t) \to x(t)$ as $k \to \infty$. Since Ω is closed, $x = x(t) \in \Omega$. Then for $t \geq \tau$ we obtain

$$\begin{aligned}
&|(S_2x_k)(t) - (S_2x)(t)| \\
&\quad = \Big| \int_t^\infty \Big(e^{-\mu\tau}[b_1(x_k(s-\tau))x_k(s-\tau) - b_2'(s-\tau)x_k(s-\tau) + b_2(s-\tau)f(x_k(s-\tau))] - f(x_k(s)) \\
&\qquad - \Big(e^{-\mu\tau}[b_1(x(s-\tau))x(s-\tau) - b_2'(s-\tau)x(s-\tau) + b_2(s-\tau)f(x(s-\tau))] - f(x(s)) \Big)\Big) ds \Big| \\
&\quad \leq \int_\tau^\infty \Big| e^{-\mu\tau}[b_1(x_k(s-\tau))x_k(s-\tau) - b_1(x(s-\tau))x(s-\tau) \\
&\qquad\qquad - b_2'(s-\tau)(x_k(s-\tau) - x(s-\tau)) + b_2(s-\tau)(f(x_k(s-\tau)) - f(x(s-\tau)))] \\
&\qquad - f(x_k(s)) + f(x(s)) \Big| ds.
\end{aligned}$$

From (2.5), (2.6) it follows that

$$\begin{aligned}
\Big| \int_\tau^\infty \Big(e^{-\mu\tau}[b_1(x(s-\tau))x(s-\tau) - b_2'(s-\tau)x(s-\tau) \\
+ b_2(s-\tau)f(x(s-\tau))] - f(x(s)) \Big) ds \Big| < \infty. \qquad (2.7)
\end{aligned}$$

Since $x_k(s-\tau) - x(s-\tau) \to 0, f(x_k(s)) - f(x(s)) \to 0$ as $k \to \infty$, by applying the Lebesgue dominated convergence theorem we get

$$\lim_{k\to\infty} \|(S_2x_k)(t) - (S_2x)(t)\| = 0.$$

This means that S_2 is continuous.

We will show that $S_2\Omega$ is relatively compact. It is sufficient to show by the Arzelà–Ascoli theorem that the family of functions $\{S_2x : x \in \Omega\}$ is uniformly bounded and equicontinuous on every finite subinterval of $[0, \infty)$. The uniform boundedness follows from the definition of Ω. For the equicontinuity we only need to show, according to Levitan's result [14], that for any given $\varepsilon > 0$ the interval $[0, \infty)$ can be decomposed into finite subintervals in such a way that on each subinterval all functions of the family have a change of amplitude less than ε. Then with regard to condition (2.7), for $x \in \Omega$ and any $\varepsilon > 0$ we choose $t^* \geq \tau$ large enough so that

$$\Big| \int_{t^*}^\infty \Big(e^{-\mu\tau}[b_1(x(s-\tau))x(s-\tau) - b_2'(s-\tau)x(s-\tau) + b_2(s-\tau)f(x(s-\tau))] - f(x(s)) \Big) ds \Big| < \frac{\varepsilon}{2}.$$

Then for $x \in \Omega, T_2 > T_1 \geq t^*$ we get

$$\begin{aligned}
&|(S_2x)(T_2) - (S_2x)(T_1)| \\
&\quad \leq \Big| \int_{T_2}^\infty \Big(e^{-\mu\tau}[b_1(x(s-\tau))x(s-\tau) - b_2'(s-\tau)x(s-\tau) + b_2(s-\tau)f(x(s-\tau))] - f(x(s)) \Big) ds \Big| \\
&\qquad + \Big| \int_{T_1}^\infty \Big(e^{-\mu\tau}[b_1(x(s-\tau))x(s-\tau) - b_2'(s-\tau)x(s-\tau) + b_2(s-\tau)f(x(s-\tau))] \\
&\qquad\qquad - f(x(s)) \Big) ds \Big| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

For $x \in \Omega$ and $\tau \le T_1 < T_2 \le t^*$ we obtain

$$\begin{aligned}
&|(S_2x)(T_2) - (S_2x)(T_1)| \\
&\quad \le \left| \int_{T_1}^{T_2} \Big(e^{-\mu\tau} \big[b_1(x(s-\tau))x(s-\tau) - b_2'(s-\tau)x(s-\tau) \right. \\
&\qquad\qquad \left. + b_2(s-\tau)f(x(s-\tau)) \big] - f(x(s)) \Big)\, ds \right| \\
&\quad \le \max_{\tau \le s \le t^*} \Big\{ \Big| e^{-\mu\tau} \big[b_1(x(s-\tau))x(s-\tau) - b_2'(s-\tau)x(s-\tau) \\
&\qquad\qquad + b_2(s-\tau)f(x(s-\tau)) \big] - f(x(s)) \Big| \Big\} (T_2 - T_1).
\end{aligned}$$

Thus there exists $\delta = \varepsilon / B$, where

$$\begin{aligned}
B = \max_{\tau \le s \le t^*} \Big\{ \Big| e^{-\mu\tau} &\big[b_1(x(s-\tau))x(s-\tau) - b_2'(s-\tau)x(s-\tau) \\
&+ b_2(s-\tau)f(x(s-\tau)) \big] - f(x(s)) \Big| \Big\},
\end{aligned}$$

such that

$$|(S_2x)(T_2) - (S_2x)(T_1)| < \varepsilon \quad \text{if } 0 < T_2 - T_1 < \delta.$$

Finally for any $x \in \Omega$, $0 \le T_1 < T_2 \le \tau$ there exists a $\delta_1 > 0$ such that

$$\begin{aligned}
&|(S_2x)(T_2) - (S_2x)(T_1)| \\
&\quad = |w(T_2) - w(T_1)| = \left| \int_{T_1}^{T_2} w'(s)\, ds \right| \\
&\quad \le \int_{T_1}^{T_2} |w'(s)|\, ds \le \max_{0 \le s \le \tau} \{|w'(s)|\}(T_2 - T_1) < \varepsilon \quad \text{if } 0 < T_2 - T_1 < \delta_1.
\end{aligned}$$

Consequently $\{S_2x : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[0, \infty)$ and hence $S_2\Omega$ is relatively compact subset of $C([0, \infty), R)$. By Lemma (1.1) there is an $x_0 \in \Omega$ such that $S_1x_0 + S_2x_0 = x_0$. Thus $x_0(t)$ is a positive solution of equation (1.6). The proof is completed. □

Corollary 2.2. *Suppose that (2.1) holds and there exist bounded functions $v, w \in C^1([0, \infty), (0, \infty))$, constant $K \ge 0$ such that (2.2), (2.4) hold and*

$$w'(t) - v'(t) \le 0, \qquad 0 \le t \le \tau. \tag{2.8}$$

Then equation (1.6) has a positive solution which is bounded by the functions v, w.

Proof. We only need to prove that the condition (2.8) implies (2.3) Let $t \in [0, \tau]$ and set

$$H(t) = w(t) - w(\tau) - v(t) + v(\tau).$$

Then with regard to (2.8) it follows that $H'(t) = w'(t) - v'(t) \le 0$, $0 \le t \le \tau$. Since $H(\tau) = 0$ and $H'(t) \le 0$ for $t \in [0, \tau]$, this implies that

$$H(t) = w(t) - w(\tau) - v(t) + v(\tau) \ge 0, \qquad 0 \le t \le \tau.$$

Thus all conditions of Theorem (2.1) are satisfied. □

Corollary 2.3. *Suppose that (2.1) holds and there exists bounded function $w \in C^1([0,\infty),(0,\infty))$, constant $K \geq 0$ such that*

$$
\begin{aligned}
b_2(t-\tau)&e^{-\mu\tau}w(t-\tau)\\
&= w(t) - K\\
&\quad + \int_t^\infty \Big(e^{-\mu\tau}\big[b_1(w(s-\tau))w(s-\tau) - b_2'(s-\tau)w(s-\tau)\\
&\qquad\qquad + b_2(s-\tau)f(w(s-\tau))\big] - f(w(s))\Big)\,ds, \quad t \geq \tau.
\end{aligned}
\tag{2.9}
$$

Then equation (1.6) has a solution $x(t) = w(t)$, $t \geq \tau$.

Proof. We put $v(t) = w(t)$ and apply Theorem (2.1). □

Theorem 2.4. *Suppose that (2.1) holds and there exist bounded functions $v, w \in C^1([0,\infty),(0,\infty))$, constant $K \geq 0$ such that (2.2)–(2.4) hold and*

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} w(t) = k \geq 0. \tag{2.10}$$

Then equation (1.6) has a positive solution which is bounded by the functions v, w and tends to k as $t \to \infty$.

Proof. The proof of Theorem (2.4) follows from Theorem (2.1) and condition (2.10). □

Corollary 2.5. *Assume that (2.1)) holds and there exists bounded function $w \in C^1([0,\infty),(0,\infty))$, constant $K \geq 0$ such that (2.9) holds and*

$$\lim_{t\to\infty} w(t) = k \geq 0.$$

Then equation (1.6) has a solution $x(t) = w(t)$, $t \geq \tau$, which tends to k as $t \to \infty$.

Proof. We set $v(t) = w(t)$ and apply Theorem (2.4). □

The following theorem shows how to construct the functions v, w to meet the conditions of Theorem (2.4).

Theorem 2.6. *Suppose that $0 < k_1 \leq k_2$, $p \in C(R, (0,\infty))$ and there exist constants $\gamma \geq 0$, $\tau > t_0 \geq 0$ such that*

$$\frac{k_1}{k_2}\exp\left((k_2 - k_1)\int_{t_0-\gamma}^{t_0} p(t)\,dt\right) \geq 1, \qquad t \geq \tau \tag{2.11}$$

$$
\begin{aligned}
&\exp\Big(-k_2\int_{t-\tau}^{t} p(s)\,ds\Big)+\exp\Big(k_2\int_{t_0-\gamma}^{t-\tau} p(s)ds\Big)\\
&\quad\times\int_t^\infty\Big(e^{-\mu\tau}\Big[b_1\Big(\exp\Big(-k_1\int_{t_0-\gamma}^{s-\tau}p(u)\,du\Big)\Big)\exp\Big(-k_1\int_{t_0-\gamma}^{s-\tau}p(u)\,du\Big)\\
&\qquad\qquad -b_2'(s-\tau)\exp\Big(-k_1\int_{t_0-\gamma}^{s-\tau}p(u)\,du\Big)\\
&\qquad\qquad +b_2(s-\tau)f\Big(\exp\Big(-k_1\int_{t_0-\gamma}^{s-\tau}p(u)\,du\Big)\Big)\Big]\\
&\qquad -f\Big(\exp\Big(-k_2\int_{t_0-\gamma}^{s}p(u)\,du\Big)\Big)\Big)ds\\
&\le b_2(t-\tau)e^{-\mu\tau}\le \exp\Big(-k_1\int_{t-\tau}^{t}p(s)ds\Big)+\exp\Big(k_1\int_{t_0-\gamma}^{t-\tau}p(s)\,ds\Big)\\
&\quad\times\int_t^\infty\Big(e^{-\mu\tau}\Big[b_1\Big(\exp\Big(-k_2\int_{t_0-\gamma}^{s-\tau}p(u)\,du\Big)\Big)\exp\Big(-k_2\int_{t_0-\gamma}^{s-\tau}p(u)\,du\Big)\\
&\qquad\qquad -b_2'(s-\tau)\exp\Big(-k_2\int_{t_0-\gamma}^{s-\tau}p(u)\,du\Big)\\
&\qquad\qquad +b_2(s-\tau)f\Big(\exp\Big(-k_2\int_{t_0-\gamma}^{s-\tau}p(u)\,du\Big)\Big)\Big]\\
&\qquad -f\Big(\exp\Big(-k_1\int_{t_0-\gamma}^{s}p(u)\,du\Big)\Big)\Big)\,ds.
\end{aligned}
\tag{2.12}
$$

Then equation (1.6) *has a positive solution.*

Proof. We set

$$
v(t)=\exp\Big(-k_2\int_{t_0-\gamma}^{t}p(s)\,ds\Big),\quad w(t)=\exp\Big(-k_1\int_{t_0-\gamma}^{t}p(s)\,ds\Big),\quad t\ge t_0.
$$

We will show that the conditions of Corollary (2.2) are satisfied. With regard to (2.8), for $t\in[t_0,\,\tau]$ we get

$$
\begin{aligned}
w'(t)-v'(t)&=-k_1p(t)w(t)+k_2p(t)v(t)\\
&=p(t)w(t)\Big[-k_1+k_2v(t)\exp\Big(k_1\int_{t_0-\gamma}^{t}p(s)ds\Big)\Big]\\
&=p(t)w(t)\Big[-k_1+k_2\exp\Big((k_1-k_2)\int_{t_0-\gamma}^{t}p(s)\,ds\Big)\Big]\\
&\le p(t)w(t)\Big[-k_1+k_2\exp\Big((k_1-k_2)\int_{t_0-\gamma}^{t}p(s)\,ds\Big)\Big]\le 0.
\end{aligned}
$$

Set $t_0=0$, $K=0$ and other conditions of Corollary 2.2 are also satisfied. The proof is completed. □

3 Examples

The following examples illustrate our results.

Example 3.1. Consider the neutral differential equation

$$
x'(t)=[b_2x'(t-\tau)+b_2f(x(t-\tau))+b_1x(t-\tau)]e^{-\mu\tau}-f(x(t)),\qquad t\ge\tau,
\tag{3.1}
$$

where

$$\tau > 0, \quad \mu = 2, \quad b_1 = (e^{0.5\tau} - 1)e^{\tau}, \quad b_2 = e^{1.5\tau}, \quad f(x) = x^{0.5}, \quad x > 0.$$

We will show that the conditions of Corollary 2.5 are satisfied. Condition

$$b_2 e^{-\mu\tau} < 1$$

obviously holds. For $K = 0$ and $w(t) = e^{-t}$ we get

$$b_2 e^{-\mu\tau} w(t-\tau) = w(t) + \int_t^\infty \Big(e^{-\mu\tau}[b_1 w(s-\tau) + b_2 f(w(s-\tau))] - f(w(s)) \Big)\, ds, \qquad t \ge \tau.$$

Then $x(t) = w(t) = e^{-t}$ is the solution of (3.1) for $t \ge \tau$ and tends to zero as $t \to \infty$. For function

$$u_0(t) = \Big[e^{-0.5(t-\tau)} - 1\Big] e^{-(t-\tau)}, \qquad 0 \le t \le \tau,$$

$x(t) = e^{-t}$ is also solution of equation (1.5) for $t \in [0, \tau]$.

Example 3.2. Consider the neutral differential equation

$$x'(t) = [b_2 x'(t-\tau) + b_2 f(x(t-\tau)) + b_1 x(t-\tau)]e^{-\mu\tau} - f(x(t)), \qquad t \ge \tau, \tag{3.2}$$

where

$$\mu > 0, \quad \tau > 0, \quad f(x) = ax, \quad a > 0, \quad b_1 = a\Big(1 - \frac{a}{r} - \frac{r-a}{r} e^{-r\tau}\Big) e^{\mu\tau},$$
$$b_2 = \Big(\frac{a}{r} + \frac{r-a}{r} e^{-r\tau}\Big) e^{\mu\tau}, \quad r > a.$$

We will show that for $w(t) = k + e^{-rt}$, $k > 0$ and $K = k\big(1 - b_2 e^{-\mu\tau}\big)$ the conditions of Corollary (2.5) are satisfied. Consequently we get

$$1 - b_2 e^{-\mu\tau} = 1 - \frac{a}{r} - \frac{r-a}{r} e^{-r\tau} = \frac{r-a}{r}\Big(1 - e^{-r\tau}\Big) > 0.$$

This implies that (2.1) holds and $K > 0$. For the condition (2.9) we get

$$\begin{aligned}
& b_2 e^{-\mu\tau} w(t-\tau) \\
&\quad = k + e^{-rt} - k\Big(1 - b_2 e^{-\mu\tau}\Big) \\
&\qquad + \int_t^\infty \Big(e^{-\mu\tau}\Big[b_1\Big(k + e^{-r(s-\tau)}\Big) + ab_2\Big(k + e^{-r(s-\tau)}\Big)\Big] - a\Big(k + e^{-rs}\Big)\Big)\, ds \\
&\quad = e^{-rt} + kb_2 e^{-\mu\tau} \\
&\qquad + \int_t^\infty \Big(b_1 k e^{-\mu\tau} + b_1 e^{-r(s-\tau)-\mu\tau} + ab_2 k e^{-\mu\tau} + ab_2 e^{-r(s-\tau)-\mu\tau} - ak - ae^{-rs}\Big)\, ds.
\end{aligned}$$

Since $b_1 e^{-\mu\tau} + ab_2 e^{-\mu\tau} - a = 0$, we obtain

$$\begin{aligned}
b_2 e^{-\mu\tau} w(t-\tau) &= e^{-rt} + kb_2 e^{-\mu\tau} + \Big(b_1 e^{(r-\mu)\tau} + ab_2 e^{(r-\mu)\tau} - a\Big) \int_t^\infty e^{-rs}\, ds \\
&= kb_2 e^{-\mu\tau} + \left(1 + \Big(b_1 e^{(r-\mu)\tau} + ab_2 e^{(r-\mu)\tau} - a\Big)\frac{1}{r}\right) e^{-rt}.
\end{aligned}$$

Since $1 + \big(b_1 e^{(r-\mu)\tau} + ab_2 e^{(r-\mu)\tau} - a\big)\frac{1}{r} = b_2 e^{(r-\mu)\tau}$, we get

$$b_2 e^{-\mu\tau} w(t-\tau) = kb_2 e^{-\mu\tau} + b_2 e^{(r-\mu)\tau} e^{-rt} = b_2 e^{-\mu\tau}\Big(k + e^{-r(t-\tau)}\Big).$$

Thus the conditions of Corollary 2.5 are satisfied and equation (3.2) has the solution $x(t) = w(t) = k + e^{-rt}$, $t \geq \tau$, such that $\lim_{t\to\infty} x(t) = k$. For function

$$u_0(t) = \Big(ak + (a-r)e^{r(t-\tau)}\Big)e^{\mu(\tau-t)}, \qquad 0 \leq t \leq \tau,$$

$$\mu > 0, \quad \tau > 0, \quad r > a > 0, \quad k \geq \frac{r-a}{a}, \quad f(x) = ax, \quad x > 0,$$

the function $x(t) = k + e^{-rt}$ is also solution of equation (1.5) for $t \in [0,\tau]$.

The applicability of Theorem 2.6 and hence also the Corollary 2.2 is illustrated in the following example.

Example 3.3. Consider the neutral differential equation

$$[x(t) - be^{-\mu}x(t-1)]' = e^{-\mu}[b_1(x(t-1))x(t-1) + bf(x(t-1))] + f(x(t)), \tag{3.3}$$

where

$$t \geq 1, \quad b \in (0,\infty), \quad \mu \in (0,\infty), \quad b_1(x) = x, \quad f(x) = x^2, \quad x > 0.$$

We will show that the conditions of Theorem 2.6 are satisfied. Condition (2.11) has a form

$$\frac{k_1}{k_2}\exp((k_2-k_1)p\gamma) \geq 1, \tag{3.4}$$

$0 < k_1 \leq k_2$, $\gamma \geq 0$, $p(t) = p \in (0,\infty)$. For condition (2.12) we get

$$\begin{aligned}
&\exp(-pk_2) + \exp(pk_2(t-1+\gamma)) \\
&\quad\times \int_t^\infty \Big(e^{-\mu}\Big[\exp(-pk_1(s-1+\gamma))\exp(-pk_1(s-1+\gamma)) \\
&\qquad\qquad + b\exp(-2pk_1(s-1+\gamma))\Big] - \exp(-2pk_2(s+\gamma))\Big)\,ds \\
&\leq b\,e^{-\mu} \\
&\leq \exp(-pk_1) + \exp(pk_1(t-1+\gamma)) \\
&\quad\times \int_t^\infty \Big(e^{-\mu}\Big[\exp(-pk_2(s-1+\gamma))\exp(-pk_2(s-1+\gamma)) \\
&\qquad\qquad + b\exp(-2pk_2(s-1+\gamma))\Big] - \exp(-2pk_1(s+\gamma))\Big)\,ds, \qquad t \geq \tau = 1.
\end{aligned}$$

We obtain

$$\begin{aligned}
&e^{-pk_2} + e^{pk_2(\gamma-1)}\left[\frac{1+b}{2pk_1}e^{-2pk_1(\gamma-1)-\mu}\,e^{p(k_2-2k_1)t} - \frac{1}{2pk_2}e^{-2p\gamma k_2}e^{-pk_2t}\right] \\
&\quad\leq be^{-\mu} \leq e^{-pk_1} + e^{pk_1(\gamma-1)}\left[\frac{1+b}{2pk_2}e^{-2pk_2(\gamma-1)-\mu}\,e^{p(k_1-2k_2)t} - \frac{1}{2pk_1}e^{-2p\gamma k_1}\,e^{-pk_1t}\right], \qquad t \geq 1.
\end{aligned}$$

For $p = b = \gamma = 1$, $t \geq 1$ we get

$$e^{-k_2} + \frac{1}{k_1}e^{(k_2-2k_1)t-\mu} - \frac{1}{2k_2}e^{-k_2(t+2)} \leq e^{-\mu} \leq e^{-k_1} + \frac{1}{k_2}e^{(k_1-2k_2)t-\mu} - \frac{1}{2k_1}e^{-k_1(t+2)}.$$

For $k_1 = 2.5$, $k_2 = 3$, $t \geq 1$ condition (3.4) is satisfied and

$$e^{-3} + \frac{1}{2.5}e^{-2t-\mu} - \frac{1}{6}e^{-3(t+2)} \leq e^{-\mu} \leq e^{-2.5} + \frac{1}{3}e^{-3.5t-\mu} - \frac{1}{5}e^{-2.5(t+2)}. \tag{3.5}$$

If μ satisfies (3.5), then equation (3.3) has a solution which is bounded by the functions $v(t) = \exp(-3(t+1))$, $w(t) = \exp(-2.5(t+1))$, $t \geq 1$. For example, $\mu = 2.8$ satisfies (3.5).

4 Conclusions

In this paper, we present a study of a nonlinear nonautonomous neutral delay differential equation, which represents a population model. This model may be suitable for describing the development of an insect population with larval and adult phases. The article is not about finding solutions of this type of equations, but about investigating the conditions for the existence of a solution. The existence of positive solutions for the equation (1.6) is treated. Let us mention that we cannot apply Theorem 4 [11] to the equation (3.2). To clarify the detailed comparison we state the mentioned theorem.

Theorem 4.1 (Theorem 4 [11], p. 4664)**.** *Suppose that $b_0 = 0, b_1 > 0$, $b_2 e^{-\mu\tau} < 1$ and that there exists $u_m^* > 0$ such that*

$$\begin{aligned} b_1 u_m e^{-\mu\tau} &> d(u_m)(1 - b_2 e^{-\mu\tau}) \quad \text{when } 0 < u_m < u_m^*, \\ b_1 u_m e^{-\mu\tau} &< d(u_m)(1 - b_2 e^{-\mu\tau}) \quad \text{when } u_m > u_m^*. \end{aligned} \tag{4.1}$$

Let $d(u_m)$ be an increasing differentiable function of u_m satisfying $d(0) = 0$ and $d(u_m) = o(u_m)$ as $u_m \to 0$. Then if $u_0(a) \in C[0,\infty), u_0(a) \geq 0$ and $u_0(a) \not\equiv 0$, then the solution of equations

$$\begin{aligned} u_m'(t) &= \big(b_2 u_m'(t-\tau) + b_2 d(u_m(t-\tau)) + b_0 u_i(t-\tau) \\ &\qquad + b_1 u_m(t-\tau)\big)e^{-\mu\tau} - d(u_m(t)), \qquad t \geq \tau, \\ u_m'(t) &= u_0(\tau - t)e^{-\mu t} - d(u_m(t)), \qquad t \leq \tau, \end{aligned}$$

satisfies $u_m(t) \to u_m^*$ as $t \to \infty$.

For $\mu > 0,\ \tau > 0,\ d(u_m) = a u_m,\ a > 0,$

$$\begin{aligned} b_1 &= a\left(1 - \frac{a}{r} - \frac{r-a}{r}e^{-r\tau}\right)e^{\mu\tau}, \\ b_2 &= \left(\frac{a}{r} + \frac{r-a}{r}e^{-r\tau}\right)e^{\mu\tau}, \qquad r > a, \end{aligned}$$

we get

$$b_1 u_m e^{-\mu\tau} = a u_m \frac{r-a}{r}\Big(1 - e^{-r\tau}\Big), \qquad d(u_m)\Big(1 - b_2 e^{-\mu\tau}\Big) = a u_m \frac{r-a}{r}\Big(1 - e^{-r\tau}\Big).$$

We have

$$b_1 u_m e^{-\mu\tau} = d(u_m)\Big(1 - b_2 e^{-\mu\tau}\Big).$$

The conditions (4.1) of Theorem (4.1) are not satisfied, but the equation (3.2) has solution $u_m = k + e^{-rt}$, $k > 0$, $t \geq \tau$ and $\lim_{t\to\infty} u_m(t) = k$. For function

$$u_0(t) = \Big(ak + (a-r)e^{r(t-\tau)}\Big)e^{\mu(\tau-t)}, \qquad 0 \leq t \leq \tau,$$

$$\mu > 0, \quad \tau > 0, \quad r > a > 0, \quad k \geq \frac{r-a}{a}, \quad d(u_m) = a u_m, \quad u_m > 0$$

the function $u_m = k + e^{-rt}$ is also solution of equation (1.5) for $t \in [0,\tau]$.

We remark that for equation (3.2) the conditions of Corollary 2.5 are satisfied and equation (3.2) has solution $x(t) = k + e^{-rt}, k > 0, t \geq \tau$ and $\lim_{t\to\infty} x(t) = k$.

In addition in this paper we consider the equations with variable coefficients and the correctness of the model is supported by the existence Theorem 2.1. As far as the authors know, there are no other existence results for the insect population model with larval and adult phases. This confirmed that the results are new.

Acknowledgements

The authors gratefully acknowledge the Scientific Grant Agency VEGA of the Ministry of Education of Slovak Republic and the Slovak Academy of Sciences for supporting this work under the Grant No. 1/0812/17. The authors would like to thank the anonymous referees for their valuable comments.

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